STUDY ON PROPERTIES OF DIVISOR GRAPHS OF FINITE GROUPS

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Abstract: Algebraic graph theory provides a natural bridge between algebra and graph theory by encoding algebraic relationships as adjacency structures. Among these, divisor graphs and coprime graphs represent two contrasting perspectives on integer interactions. In this paper, we investigate the structural properties of the divisor graph D_n, whose vertices correspond to the integers $\{1,2,3,\ldots,n\}$ and where adjacency is defined by divisibility. Fundamental graph invariants of D_n, including connectivity, diameter, vertex degree distribution, and planarity, are examined in detail. We prove that D_n is connected for all$n \geq 1$, establish an upper bound on its diameter, identify vertex as the unique maximum-degree vertex, and characterize the precise range of $n$for which D_n is planar.

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Introduction

Algebraic graph theory studies the interaction between graph structure and algebraic systems, using algebraic methods to analyze symmetry, connectivity, and other structural invariants of graphs. Within this framework, graphs constructed from algebraic objects—such as groups, rings, and semigroups—have attracted considerable attention due to their ability to encode algebraic relations into combinatorial form. Integer-based graphs defined through divisibility or coprimality relations form a notable subclass of such constructions, offering a natural setting in which algebraic properties influence graph-theoretic behavior.

Among these constructions, **divisor graphs** and **coprime graphs** arise from fundamental multiplicative properties of the integers. In a divisor graph, vertices correspond to positive integers and adjacency is defined by divisibility, producing a graph with a pronounced hierarchical structure. Coprime graphs, by contrast, connect vertices whose greatest common divisor is equal to one, emphasizing algebraic independence rather than dependence. Although these two graph families are defined by distinct adjacency conditions, they are closely related from an algebraic perspective and often exhibit complementary structural characteristics.

The algebraic structure underlying divisor graphs has a significant impact on classical graph invariants. The existence of a universal element corresponding to the identity of multiplication, namely the integer 111, induces high connectivity and strongly influences parameters such as diameter and degree distribution. At the same time, interactions among composite integers generate increasingly complex subgraphs as the order of the graph increases, raising nontrivial questions about planarity and forbidden subgraph configurations. These features make divisor graphs a natural object of study within algebraic graph theory.

In this paper, we investigate the **divisor graph** $\boldsymbol{D_n}$ with vertex set $\{1,2,\ldots n\}$ focusing on its fundamental structural properties. We establish results on connectivity, diameter, and vertex degrees, and we provide a complete characterization of the planarity of D_n. In addition, we examine the relationship between divisor graphs and coprime graphs, demonstrating how their adjacency rules reflect contrasting algebraic relations while still revealing a complementary interplay in graph structure.

By situating divisor graphs within the context of algebraic graph theory, this work contributes to a deeper understanding of how algebraic relations among integers shape graph topology. The results also support broader efforts to classify integer-based graphs according to their algebraic origins and to relate seemingly distinct graph families through shared structural principles.

Divisor graphs

The divisor graph is obtained by considering integers as vertices to connect two different vertices if one is divisible by another. Despite the fact that the adjacency condition for divisor graphs is unlike coprime graphs, there is a strong connection between coprime graphs and divisor graphs based on

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complementary as well as relative grid configurations. In general, coprime graphs and divisor graphs can often offer complementary views of integer relations—whereas coprime graphs focus on independence, divisor graphs underscore hierarchy and dependence. To systematically compare two graph types to discern their common structural characteristics along with points of departure in graph connectivity, degree, and graph Planarity is helpful for systematically varifying coprime graph variants as special instances of arithmetic graphs. Figure 3 illustrates the Divisor graph D_6.

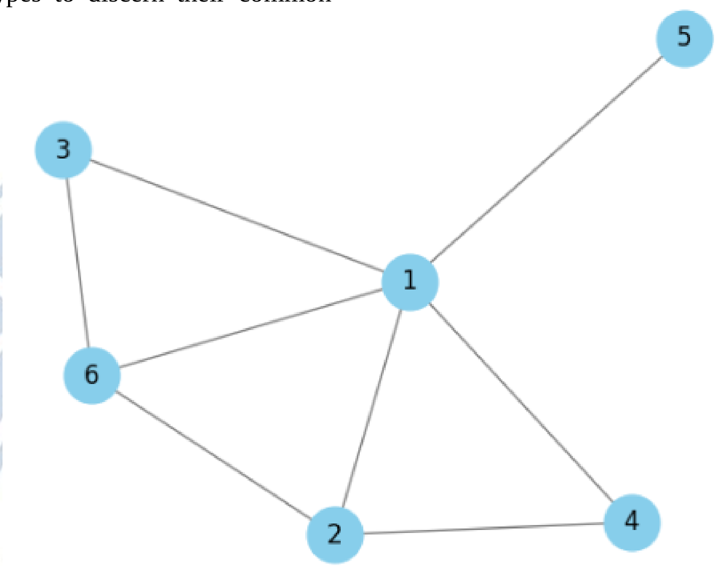

Figure 3: Divisor graph D_6

Properties:

- Vertices represent integers $\{1,2,\ldots,n\}$.
- Two vertices are adjacent if one divides the other.
- Vertex 1 is adjacent to all vertices.
- The graph often has a hierarchical or star-like structure.
- Divisor graphs are closely related to coprime graphs through complementary properties.

Theorem 1 (Connectivity of D_n)

Statement: Let D_n denote the divisor graph on $n$vertices, where vertex $i$is connected to vertex $j$if either $i \mid j$or $j \mid i$. Then for all integers $n \geq 1$, the graph $D_n$is connected. This is because vertex 1divides every other vertex in the graph, acting as a central hub, ensuring a path exists between any two vertices through vertex 1.

Proof:

Consider any two distinct vertices $u, v \in D_n$. There are two possibilities:

- If $u \mid v$or $v \mid u$, then $u$and $v$are adjacent, and the distance between them is 1.
- If neither divides the other, then vertex 1is always adjacent to both $u$and v, since $1 \mid u$and $1 \mid v$. Therefore, there exists a path of length 2:

$$u - 1 - v.$$

Since u and v were arbitrary, every pair of vertices is connected either directly or via vertex 1. Thus, D_n is connected for all $n \geq 1$. Vertex 1serves as the central hub of the graph, which not only guarantees connectivity but also heavily influences diameter and spectral properties.

Theorem 2 (Diameter of D_n)

Statement: The diameter of D_n, defined as the largest distance between any two vertices, satisfies $\operatorname{diam}(D_n) \leq 3$for $n \geq 2$. This is a consequence of the hierarchical nature of divisibility chains: integers can be connected either directly via divisibility or through intermediate factors, often involving vertex 1.

Proof: Let u and v be arbitrary vertices in D_n. Consider the following possibilities:

- Direct divisibility: If $u \mid v$or $v \mid u$, the distance is 1.
- Single intermediate vertex: If there exists a vertex $w$such that $u \mid w \mid v$or $v \mid w \mid u$, then the distance is 2.
- Indirect connection via 1: If no such intermediate vertex exists, vertex 1can connect to both $u$and v, producing a path of length 2:

$$u - 1 - v$$

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If another hierarchical step is needed, the maximum path length can reach 3. Thus, any pair of vertices in $D_n$can be connected by a path of length at most 3, giving the stated bound on the diameter. The diameter is small relative to $n$due to the central role of 1 and the multiplicative structure of integers, which produces natural hierarchical chains. Theorem 3 (Maximum Degree in D_n)

Theorem 3

Statement: In the divisor graph D_n, vertex 1has degree $n-1$, making it the central hub. Other vertices have degrees determined by the number of their divisors and multiples within $\{1,2,\dots,n\}$.

Proof:

- Vertex 1: By definition, vertex 1divides every integer $2 \leq k \leq n$, giving $n-1$edges incident to vertex 1. Therefore,

$$\deg(1) = n - 1.$$

For a general vertex $v$in the divisor graph D_n, its degree $\deg(v)$is determined by the number of vertices it is connected to. In D_n, a vertex $u$is adjacent to $v$if either $u \mid v$or $v \mid u$.

- The divisors of v(other than $v$itself) contribute edges connecting $v$to smaller numbers.
- The multiples of v(other than $v$itself) contribute edges connecting $v$to larger numbers up to n.

Formally, the degree of $v$can be written as:

$$\deg(v) = \underbrace{\#\{u \in \{1,2,\dots,n\} : u \mid v, u \neq v\}}_{\text{divisors of } v \text{ excluding } v} + \underbrace{\#\{u \in \{1,2,\dots,n\} : v \mid u, u \neq v\}}_{\text{multiples of } v \text{ excluding } v}.$$

This formula captures all edges incident to $v$in $D_n$by counting both its divisors and multiples (excluding the vertex itself).

Since vertex 1is connected to all other vertices, it is the maximum-degree vertex in D_n, acting as the main hub connecting all hierarchical divisibility chains. This property is essential for explaining diameter, connectivity, and spectral properties of the divisor graph.

Theorem 4 (Planarity Condition of D_n)

Statement: The divisor graph D_n is planar if and only if $n \leq 6$. For larger n, hierarchical divisibility chains and cross-connections among multiples may produce subgraphs homeomorphic to non-planar graphs, such as K_5 or $K_{3,3}$, making D_n non-planar.

Proof:

- Small $n \leq 6$: When $n \leq 6$, the vertex set is small, and edges form hierarchical chains and simple star connections (e.g., edges from 1 to all vertices). These can be drawn in a plane without edge crossings. No combination of divisibility connections forms a subgraph homeomorphic to K_5or $K_{3,3}$, satisfying planarity.
- Large $n > 6$: As $n$grows, multiple vertices share divisors and multiples that are not consecutive. For example, for $n = 8$, the vertices $\{2, 3, 4, 6, 8\}$form overlapping divisibility chains, producing cycles and cross edges. These can lead to the presence of subgraphs homeomorphic to K_5or $K_{3,3}$, which are non-planar by Kuratowski's theorem. Therefore, planarity fails for $n > 6$.

Planarity in D_n is directly constrained by the density of divisibility relations. The small numbers allow simple star-like or tree-like layouts, while larger numbers quickly introduce unavoidable edge crossings.

Theorem 5 (Relation Between D_n and G_n)

Statement: For any integer $n \geq 1$, the standard coprime graph G_n and the divisor graph D_n are complementary in a generalized sense. That is, if two vertices $i$and $j$are not adjacent in G_n(i.e., $\gcd(i,j) \neq 1$), they may be connected in $D_n$via a divisibility relation $i \mid j$or $j \mid i$. Conversely, adjacency in $G_n$excludes direct divisibility in most cases, illustrating a complementary structural relationship.

Proof:

1. Definition: Recall that:
    - i and j are adjacent in $G_n$if $\gcd(i,j) = 1$.
    - i and j are adjacent in $D_n$if $i \mid j$or $j \mid i$.
2. Complementary Observations:
    - If i and j are coprime, then $i \nmid j$and $j \nmid i$(unless one is 1). Therefore, adjacency in $G_n$generally implies no edge in D_n, except for vertex 1.
    - If i and j share a nontrivial common factor, they may or may not be connected in G_n. However, if one divides the other, an edge exists in D_n.

This relationship highlights that $G_n$captures mutual independence (coprimality), while $D_n$captures hierarchical dependence (divisibility). In this sense, the graphs are complementary: one emphasizes

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independence, the other emphasizes factorization chains. The interplay between G_n and D_n allows a deeper understanding of integer structure: vertices not connected in one graph may be connected in the other, and vice versa, except for the universal hub vertex 1.

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